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1988 J. Phys. A: Math. Gen. 21 1001

(http://iopscience.iop.org/0305-4470/21/4/026)

J. Phys. A: Math. Gen. 21 (1988) 1001–1015. Printed in the UK

Short-time transition probability in phase space for the Boltzmann–Fokker–Planck equation and equilibrium

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Received 29 June 1987

Abstract. We introduce the short-time transition probability associated with the most general Boltzmann–Fokker–Planck equation in cartesian coordinates, thereby extending Chandrasekhar's results. We exploit the covariant formulation of the short-time propagator for the Fokker–Planck equation as introduced by Graham and Deininghauss. The case of a locally isotropic velocity background distribution is investigated and the conditions for thermodynamic equilibrium are found. The results are a direct generalisation of Chandrasekhar's, with the covariant drift introduced by Graham replacing the ordinary drift in Chandrasekhar's simpler approach.

1. Introduction

In his pioneering work in the 1940s, Chandrasekhar was dealing with the dynamics of particles under the action of both an external field of force and Brownian stochastic forces [1,2]. He presented the transition probability to change the particle position in phase space by the amount $(\Delta \boldsymbol{v}, \Delta \boldsymbol{r})$ with a short time, where $\boldsymbol{v}$ refers to the velocity and $\boldsymbol{r}$ to the spatial coordinate. This transition probability has the structure of a transition probability in velocity subspace at a given space point, multiplied by a delta function, corresponding to advance of the particle in space by its velocity times the time interval τ. The stochastic behaviour shows itself through the transition probability in the velocity subspace. From such structure of the transition probability it is possible to derive a Boltzmann–Fokker–Planck equation (BFPE). The transition probability was obtained in those works only for the restricted case of a diagonal diffusion matrix $Q^{mn} = Q(v)\delta^{mn}$, where v is the magnitude of the particle velocity.

Chandrasekhar also analysed the conditions under which the Maxwell–Boltzmann distribution function is a static solution of the BFPE. For the diagonal diffusion matrix $Q(v)\delta^{\nu\mu}$, he found the relation $\boldsymbol{K} = \beta \boldsymbol{v} Q(v)$ where $\boldsymbol{K}$ is the drift and $\beta = 1/kT$. Chandrasekhar was aware that his analysis was not the most general and that its generalisation was yet to be obtained.

Onsager and Machlup [3] developed a path integral formulation for the solutions of the Fokker–Planck (FP) equation in the approximation of linear drift and a constant diagonal diffusion matrix. Since the middle of the 1970s much work [4–12] has been done to go beyond the case of linear drift and a constant, diagonal diffusion matrix. It is in this framework is that we have found the basic tools to complete Chandrasekhar's analysis.

There are several ways of introducing the Green function for the FP equation and its explicit expression for short times (called the short-time propagator in modern

0305-4470/88/041001+15$02.50

terminology). We have based our analysis on the covariant formulation of the short-time propagator of Deininghauss and Graham (DG) [6]. The functional solution of Graham [4, 5] is based on taking Q_{mn}, the inverse of the diffusion matrix Q^{mn}, as the metric tensor (g_{mn}). In parallel to the building of the covariant propagator, Graham formulated a covariant form of the Fokker–Planck equation based on the same metric. It is in this DG covariant framework that the general expression for the short-time transition probability in the velocity subspace is introduced. In § 2, we then build up the transition probability for phase space in a way analogous to that of Chandrasekhar.

In this work we also present a detailed treatment for the case of a locally isotropic velocity distribution function. Such a situation is relevant for the equilibrium Maxwell-Boltzmann distribution. We examine in detail the equilibrium conditions in the framework of the covariant short-time propagator (§ 4) and the covariant FP equation (§ 5) and relate them to the non-covariant expressions; we show that the equilibrium conditions obtained by these various approaches in the velocity subspace are equivalent. It is also found in the framework of the covariant formalism that the equilibrium condition in cartesian coordinates has the structure of a proportionality between $Q_{\|}$ (the diffusion in the direction parallel to the particle motion) and the modulus of the covariant drift. This condition may be considered as a generalisation of Chandrasekhar's earlier work obtained by exploiting Graham's formalism to the case of a locally isotropic velocity subspace. The formalism is introduced for cartesian velocity coordinates only. Its covariant nature and the connection with the alternative covariant formalism of Rosenbluth *et al* [13] are analysed by us in a separate paper [14]. In § 6 we present a summary and a discussion, briefly examining alternative short-time propagators to the DG formalism which we have employed.

2. The general short-time transition probability in phase space

We shall consider a system of equal mass particles moving in six-dimensional phase space with cartesian spatial coordinates $r^i (i = 1, 2, 3)$ and corresponding velocity coordinates $v^i = \mathrm{d}r^i/\mathrm{d}t$, where t denotes the time [15]†. We assume that the forces acting on a particle may be separated into an external field part (including a self-consistent field) $\boldsymbol{B}(\boldsymbol{r})$ and a fluctuating part which depends on $\boldsymbol{v}$. Consider a time interval τ that is long compared with the fluctuating timescale, but short compared to appreciable changes in particle velocity. In the interval τ, a particle at $(\boldsymbol{r}, \boldsymbol{v})$ at time t undergoes changes Δr^i in position and Δv^i in velocity given by

$$\Delta r^i = v^i \tau \tag{1}$$

$$\Delta v^i = B^i(t, \boldsymbol{r})\tau + \Delta_{\mathrm{F}} v^i \tag{2}$$

where $\Delta_{\mathrm{F}} v^i$ denotes the change in v^i due to the fluctuating part of the forces. Assume that $\Delta_{\mathrm{F}} v^i$ is associated with a conditional probability distribution

$$\chi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta r}, \boldsymbol{\Delta v}, \tau) = \Psi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta v}, \tau)\delta(\boldsymbol{\Delta r} - \boldsymbol{v}\tau) \tag{3}$$

defined such that $\chi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta r}, \boldsymbol{\Delta v}, \tau)$ is the probability that a particle at $(\boldsymbol{r}, \boldsymbol{v})$ at time t will be found at $(\boldsymbol{r} + \boldsymbol{\Delta r}, \boldsymbol{v} + \boldsymbol{\Delta v})$. The six-dimensional phase space distribution

† We follow closely the analysis found in [2, 16].

function f has to obey the equation:

$$f(t+\tau, \boldsymbol{r}+\boldsymbol{v}\tau, \boldsymbol{v}) = \int f(t, \boldsymbol{r}, \boldsymbol{v}-\boldsymbol{\Delta v})\Psi(\boldsymbol{r}, \boldsymbol{v}-\boldsymbol{\Delta v}, t; \boldsymbol{\Delta v}, \tau)\,\mathrm{d}(\boldsymbol{\Delta v}). \tag{4}$$

Expanding $f(\boldsymbol{r}+\boldsymbol{v}\tau, \boldsymbol{v}, t+\tau)$, $f(t, \boldsymbol{r}, \boldsymbol{v}-\boldsymbol{\Delta v})$ and $\Psi(\boldsymbol{r}, \boldsymbol{v}-\boldsymbol{\Delta v}, t; \boldsymbol{\Delta v}, \tau)$ in a Taylor series, we obtain

$$\frac{\partial f}{\partial t}+v^i\frac{\partial f}{\partial x^i} = -\frac{(\partial\langle\Delta v^i\rangle f)}{\partial v^i}+\frac{1}{2}\frac{\partial^2(\langle\Delta v^i\Delta v^j\rangle f)}{\partial v^i\,\partial v^j} \tag{5}$$

where the averages on the right-hand side of (5) are given by

$$\langle\Delta v^i\rangle = \frac{1}{\tau}\int \Psi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta v}, \tau)\boldsymbol{\Delta v}^i\,\mathrm{d}(\boldsymbol{\Delta v}) \tag{6a}$$

$$\langle\Delta v^i\Delta v^j\rangle = \frac{1}{\tau}\int \Psi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta v}, \tau)\Delta v^i\Delta v^j\,\mathrm{d}(\boldsymbol{\Delta v}). \tag{6b}$$

While carrying out the expansion we have neglected terms of order greater than τ and also moments of order three, like $(1/\tau)\int\Psi\Delta v^i\Delta v^j\Delta v^k\,\mathrm{d}(\Delta v)$ and higher.

The specific structure of Ψ is consistent with the neglect of those higher moments due to their being of order greater than one in τ. However, we must remember that this basic neglect of higher-order moments must be anchored in physical reasoning. When the explicit form of Ψ is introduced in § 3, it is shown that the introduction of the external force does not affect the second moment as it appears in (6*b*) relative to the situation where only fluctuating forces are present [17]†. So $\langle\Delta v^i\Delta v^j\rangle$ may be replaced by $\langle\Delta_\mathrm{F}v^i\Delta_\mathrm{F}v^j\rangle$ which is defined to be the diffusion coefficient Q^{ij}:

$$Q^{ij} = \langle\Delta_\mathrm{F}v^i\Delta_\mathrm{F}v^j\rangle = \frac{1}{\tau}\int \Psi(\boldsymbol{r}, \boldsymbol{v}, t; \boldsymbol{\Delta v}, \tau)\Delta_\mathrm{F}v^i\Delta_\mathrm{F}v^j\,\mathrm{d}(\boldsymbol{\Delta v}). \tag{7}$$

It would also appear that $\langle\Delta v^i\rangle$ is decomposed in the presence of a force $\boldsymbol{B}$ according to the equation‡

$$\langle\Delta v^i\rangle = B^i + K^i \tag{8}$$

where

$$K^i = \frac{1}{\tau}\int \Psi(r, v, t; \Delta v, \tau)\Delta_\mathrm{F}v^i\,\mathrm{d}(\Delta v) \tag{9}$$

represents the first moment, called the drift in the absence of a force. Using (7) and (8) and rearranging terms in (5) we get the Boltzmann–Fokker–Planck equation

$$\frac{\partial f}{\partial t}+v^i\frac{\partial f}{\partial x^i}+B^i\frac{\partial f}{\partial v^i} = -\frac{\partial(K^if)}{\partial v^i}+\frac{1}{2}\frac{\partial(Q^{ij}f)}{\partial v^i\,\partial v^j}. \tag{10}$$

This equation is denoted the Boltzmann–Fokker–Planck equation, since the right-hand side of (10) gives the form of the Fokker–Planck scattering contribution to the collision term, while the left-hand side has the form of the comoving time derivative

$$\mathrm{D}f/\mathrm{D}t = \partial f/\partial t + v^i\,\partial f/\partial x^i + B_i\,\partial f/\partial v^i = (\partial f/\partial t)_{\text{collisions}}. \tag{11}$$

† See [2], pp 31–5.

‡ This analysis is consistent with equation (2) and would appear to be also consistent within the framework of the short-time transition probability in § 4.

We shall follow the formulation of Deininghauss and Graham [6] to obtain the short-time transition probability for the velocity subspace in accordance with the general FP equation.

Next we add a force term. Combining this with (3) leads to a short-time transition probability consistent with the general BFPE, equation (10),

$$\frac{\mathrm{D}f(\boldsymbol{r},\boldsymbol{v})}{\mathrm{D}t}=\frac{(K^{\nu}(\boldsymbol{r},\boldsymbol{v})f(\boldsymbol{r},\boldsymbol{v}))}{\partial v^{\nu}}+\frac{1}{2}\frac{\partial^2(Q^{\mu\nu}(\boldsymbol{r},\boldsymbol{v})f(\boldsymbol{r},\boldsymbol{v}))}{\partial v^{\mu}\,\partial v^{\nu}}. \tag{12}$$

From now on when the discussion is limited to the velocity subspace we will omit the r dependence.

The conditional probability function in velocity space, $P(\boldsymbol{v};\boldsymbol{v}_0,t)$, for having velocity $\boldsymbol{v}$ at time t after the particle had initial velocity $\boldsymbol{v}_0$ satisfies the same FP equation as for f:

$$\frac{\partial P(\boldsymbol{v};\boldsymbol{v}_0,t)}{\partial t}=-\frac{\partial(K^{\nu}(\boldsymbol{v})P(\boldsymbol{v},\boldsymbol{v}_0,t))}{\partial v^{\nu}}+\frac{1}{2}\frac{\partial^2 Q^{\nu\mu}(\boldsymbol{v})P(\boldsymbol{v},\boldsymbol{v}_0,t)}{\partial v^{\nu}\,\partial v^{\mu}}. \tag{13}$$

Graham [4, 5] derived a covariant path integral representation of (13). His formalism is a covariant one based on choosing the inverse of the diffusion matrix $Q^{\nu\mu}(v)$ as the metric tensor in the problem. We shall emphasise this role by introducing the notation

$$g_{\mu\nu}\equiv Q_{\mu\nu}\qquad g^{\mu\nu}\equiv Q^{\mu\nu}\qquad Q^{\mu\nu}Q_{\nu\lambda}=\delta^{\mu}_{\lambda}. \tag{14}$$

This choice of the metric is possible due to the fact that the diffusion matrix $Q^{\mu\nu}(v)$ is a positive definite symmetric tensor. In terms of these metrical concepts the path integral representation of (13) becomes

$$P(\boldsymbol{v},\boldsymbol{v}_0,t)=\int \mathrm{D}\mu(\{\boldsymbol{v}\})\exp\left(-\int L(\dot{\boldsymbol{v}},\boldsymbol{v})\,\mathrm{d}\tau\right) \tag{15}$$

(postpoint) with the Lagrangian

$$L(\dot{\boldsymbol{v}},\boldsymbol{v})=\frac{1}{2}g_{\mu\nu}(\dot{v}^{\mu}-h^{\mu})(\dot{v}^{\nu}-h^{\nu})-\frac{1}{2\sqrt{g}}\frac{\partial(h^{\nu}\sqrt{g})}{\partial v^{\nu}}+\frac{R}{12} \tag{16}$$

and the measure of the integration is

$$\begin{aligned}\mathrm{D}\mu(\{\boldsymbol{v}\})&=\lim_{\tau\to 0}\prod_{j=1}^{N-1}\frac{\mathrm{d}v(\tau_j)}{[(2\pi\tau)^3g^{-1}(v(\tau_j))]^{1/2}[(2\pi\tau)^3g^{-1}(\boldsymbol{v})]^{1/2}}\\ \mathrm{d}v&=\prod_{\nu=1}^{3}\mathrm{d}v^{\nu}\end{aligned} \tag{17}$$

where

$$h^{\nu}=K^{\nu}-\frac{1}{2\sqrt{g}}\frac{\partial(\sqrt{g}g^{\nu\mu})}{\partial v^{\mu}} \tag{18}$$

defines the covariant drift h^{ν} and we also have the following definitions:

$$g(\boldsymbol{v})=\det(g_{\mu\nu}) \tag{19a}$$

$$R=g^{\nu\lambda}g^{\mu\kappa}R_{\nu\mu\lambda\kappa} \tag{19b}$$

where $R_{\nu\mu\lambda\kappa}$ is the Riemann curvature tensor.

We can immediately pass from the FP equation (13) for the conditional probability to the FP equation (12) for the distribution function. Given that

$$f(\boldsymbol{v}, t) = \int P(\boldsymbol{v}, \boldsymbol{v}_0, t) f(\boldsymbol{v}_0)\, \mathrm{d}\boldsymbol{v}_0 \tag{20}$$

then by multiplying (13) with $f(\boldsymbol{v}_0)$ and integrating over $\boldsymbol{v}_0$, we get (12) if we use the fact that the time derivative and the velocity derivatives at the postpoint $\boldsymbol{v}$ commute with the prepoint coordinate v_0. After introducing the general form of the propagator, DG derive the expression for the short-time propagator in which the various functions are evaluated at the postpoint $\boldsymbol{v}$.

For short times DG point out that the naive short-time propagator obtained by taking just one discrete interval out of (15),

$$A_\tau(\boldsymbol{v}, \boldsymbol{v}_0) = \sqrt{g(\boldsymbol{v})}(2\pi\tau)^{-3/2} \exp\{-\tau L[(\boldsymbol{v}-\boldsymbol{v}_0)/\tau, \boldsymbol{v}]\} \tag{21}$$

must be corrected to be a consistent expression for a short-time propagator. The short-time propagator was then obtained by multiplying this rough approximation by a correction factor $z(\boldsymbol{v}-\boldsymbol{v}_0, \boldsymbol{v}, \tau)$

$$P(\boldsymbol{v}, \boldsymbol{v}_0, \tau) = A_\tau(\boldsymbol{v}, \boldsymbol{v}_0) z(\boldsymbol{v}-\boldsymbol{v}_0, \boldsymbol{v}, \tau). \tag{22}$$

The correction function z has the form of a finite power series in $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{v}_0$, whose coefficients are determined such that the short-time propagator satisfies the Chapman-Kolmogorov equation:

$$P(\boldsymbol{v}, \boldsymbol{v}_0, 2\tau) = \int \mathrm{d}\boldsymbol{v}_1 P(\boldsymbol{v}, \boldsymbol{v}_1, \tau) P(\boldsymbol{v}_1, \boldsymbol{v}_0, \tau). \tag{23}$$

The condition, also known as the fixed point condition, means that if we multiply two consecutive short-time propagators and integrate over the intermediate point v_1, we get the short-time propagator for the double time interval 2τ. So the final covariant explicit form of the short-time propagator in (22) is now:

$$\begin{aligned} P(\boldsymbol{v}, \boldsymbol{v}_0, \tau) = \bigg(& 1 + C_{\alpha\beta}(\boldsymbol{v})\eta^\alpha\eta^\beta + \frac{1}{\tau}(D_{\alpha\beta\gamma}(\boldsymbol{v})\eta^\alpha\eta^\beta\eta^\gamma + E_{\alpha\beta\gamma\delta}(v)\eta^\alpha\eta^\beta\eta^\gamma\eta^\delta) \\ & + \frac{1}{\tau^2} G_{\alpha\beta\gamma\delta\varepsilon\nu}(\boldsymbol{v})\eta^\alpha\eta^\beta\eta^\gamma\eta^\delta\eta^\varepsilon\eta^\nu \bigg) \\ & \times \bigg(\frac{1}{(2\pi\tau)^{3/2}} \sqrt{g(\boldsymbol{v})} \exp[-\tfrac{1}{2}\tau g_{\mu\nu}(\boldsymbol{v})(\eta^\mu/\tau - h^\mu)(\eta^\nu/\tau - h^\nu) \\ & - \tfrac{1}{2}\tau(h^\nu{}_{;\nu}(\boldsymbol{v})) + \tfrac{1}{6}R(\boldsymbol{v})] \bigg). \end{aligned} \tag{24}$$

The various coefficients are given by

$$C_{\alpha\beta} = -\tfrac{1}{12}R_{\alpha\beta} - \tfrac{1}{2}(\partial(g_{.\nu}h^\nu)/\partial v^{\cdot}]_{\alpha\beta} \tag{25a}$$

where $R_{\alpha\beta}$ is the Ricci tensor, and

$$D_{\alpha\beta\delta} = \tfrac{1}{4}\{\partial g_{..}/\partial v^{\cdot}\}_{\alpha\beta} \tag{25b}$$

$$E_{\alpha\beta\gamma\delta} = \tfrac{1}{12}\{\partial^2 g_{..}/\partial v^{\cdot}\,\partial v^{\cdot} - \tfrac{1}{2}g^{\nu\mu}\Gamma^{\nu}_{..}\Gamma^{\mu}_{..}\}_{\alpha\beta\gamma\delta} \tag{25c}$$

$$G_{\alpha\beta\gamma\delta\varepsilon\nu} = \tfrac{1}{32}\{\partial g_{..}/\partial v^{\cdot}\,\partial g_{..}/\partial v^{\cdot}\}_{\alpha\beta\gamma\delta\varepsilon\nu}. \tag{25d}$$

The curly brackets denote complete symmetrisation, e.g.

$$\{\partial g_{..}/\partial v^{\cdot}\}_{\alpha\beta\gamma} = \tfrac{1}{3}(\partial g_{\alpha\beta}/\partial v^\gamma + \partial g_{\alpha\gamma}/\partial v^\beta + \partial g_{\beta\gamma}/\partial v^\alpha).$$

The expression in equation (24) may also be derived in terms of a WKB approximation of equation (15) [6]. It has the form of a scalar density which consists of a scalar part S multiplied by $\sqrt{g}$. When integrated over velocity we must multiply only $\mathrm{d}\boldsymbol{v} = \mathrm{d}v^1\,\mathrm{d}v^2\,\mathrm{d}v^3$, so that the integral element has the form of $S\sqrt{g}\,\mathrm{d}\boldsymbol{v}$, as it should have in curved space formalism. The time-dependent structure of the short-time propagator means that the FP equation (12) and the Chapman–Kolmogorov condition are satisfied up to order τ.

In equation (24) the short-time propagator has a complicated form. By passing to Riemannian normal coordinates [15] at the postpoint, a much simpler form of the propagator is obtained. Denoting by a prefactor $*$ all quantities expressed in normal coordinates, the short-time propagator becomes

$$P(\boldsymbol{v}, {}^*\boldsymbol{v}, \tau) = (2\pi\tau)^{-3/2}\sqrt{g(\boldsymbol{v})}[1 + {}^*C_{\alpha\beta}(\boldsymbol{v})\,{}^*\eta^{\alpha}\,{}^*\eta^{\beta} + \ldots]$$

$$\times \exp\left[\frac{\tau^* g_{\nu\mu}(\boldsymbol{v})}{2}\left(\frac{{}^*\eta^{\nu}}{\tau} - {}^*h^{\nu}(\boldsymbol{v})\right)\left(\frac{{}^*\eta^{\mu}}{\tau} - {}^*h^{\mu}(\boldsymbol{v})\right) - \frac{\tau}{2}\left(\frac{\partial^* h^{\nu}}{\partial v^{\nu}} + \frac{{}^*R}{6}\right)\right] \tag{26}$$

where

$${}^*C_{\alpha\beta} = -\tfrac{1}{2}\{{}^*g_{\nu'}\,\partial^* h^{\nu'}/\partial v^{\cdot}\}_{\alpha\beta} - \tfrac{1}{2}{}^*R_{\alpha\beta}. \tag{27}$$

We will now introduce a force term $\boldsymbol{B}(\boldsymbol{r})$, which as we have noted can include an internal self-consistent force as well as an external force. In the presence of such a force, we must add the systematic change $\langle\Delta\boldsymbol{v}\rangle/\tau$ due to that force into the drift term, and thus K^{ν} is replaced by $K^{\nu} + B^{\nu}$ and, in accordance with equations (2) and (8), we define

$$\tilde{h}^{\nu} = h^{\nu}(\boldsymbol{r}, \boldsymbol{v}) + B^{\nu}(\boldsymbol{r}). \tag{28}$$

To make this change consistent with the covariant nature of the velocity subspace transformations, $B^i(\boldsymbol{r})$ must transform like a contravariant vector.

All the elements to construct the short-time transition probability for phase space have now been presented. In analogy with Chandrasekhar's prescription of (3), we write the final expression for the short-time conditional probability:

$$P(\boldsymbol{r}, \boldsymbol{v}; \boldsymbol{r}_0, \boldsymbol{v}_0, \tau) = P_v(\boldsymbol{r}_0, \boldsymbol{v}; \boldsymbol{v}_0, \tau)\,\delta(\boldsymbol{r} - \boldsymbol{r}_0 - \boldsymbol{v}\tau) \tag{29}$$

where P_v is the conditional probability in the velocity subspace located at $\boldsymbol{r}_0$ and where we have replaced the Ψ notation appearing in (3) with the conditional transition probability of (24), where the relevant differentials $\Delta\boldsymbol{v}, \Delta\boldsymbol{r}$ are defined to be

$$\Delta\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0 \tag{30a}$$

$$\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0. \tag{30b}$$

The basic structure of Chandrasekhar's transition probability is retained to the DG expression of (24) for the velocity subspace transition probability and with the inclusion of a force term in the drift through (26). The reconstruction of the FP equation out of (27) will be shown in detail in § 4. It must be noted that when making a transformation within the velocity subspace, the $\boldsymbol{v}$ term appearing in the delta function must retain its vectorial properties. This means that it has the same magnitude and direction relative to the non-curved spatial space $\boldsymbol{r}$. Otherwise, the particles will reach a different place due to transformations in the velocity subspace, in contradiction with (1).

3. The structure of curved velocity space for a locally isotropic background

In the last section the general formula for the short-time propagator in the velocity subspace was presented (equations (24)-(26)). Our main interest is in working out the details of the locally isotropic velocity with a view to analysing the conditions of equilibrium, in §§ 4 and 5.

The situation is much simplified when we are dealing with the case of locally isotropic functions in velocity space. Working with locally cartesian coordinates in velocity space we can diagonalise the diffusion matrix and its inverse. Let the v^1 component of the velocity be in the direction of the velocity vector, i.e. v^2 and v^3 are zero. Defining

$$Q_{\parallel} = Q^{11}(\boldsymbol{v}) \tag{31a}$$

and

$$Q_{\perp} = Q^{22}(\boldsymbol{v}) = Q^{33}(\boldsymbol{v}) \tag{31b}$$

while off-diagonal components of Q^{ij} vanish, we observe that

$$K^{i} = \eta(\boldsymbol{v})\delta^{1'}. \tag{32}$$

While introducing our notation $g_{ij} = Q_{ij}$ and hence

$$g = \det(g_{ij}) \tag{33}$$

we also define the quantity $\tilde{Q}$ for convenient comparison with other work:

$$\tilde{Q} = \det(Q^{ij}) \tag{34}$$

whence $\tilde{Q} = g^{-1} = Q_{\parallel}Q_{\perp}^{2}$. In these equations all the quantities appear to be functions of the magnitude of the velocity alone. The general form of the diffusion matrix is obtained by rotating using Euler angles (figure 1) from the direction parallel to the

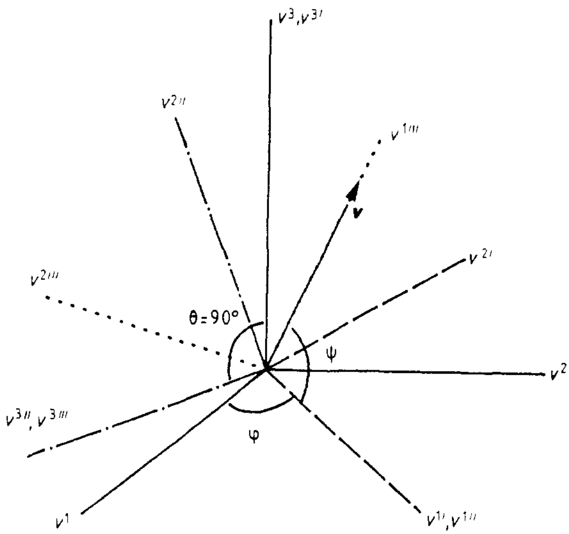

Figure 1. The transformation of the velocity coordinates v^i into $v^{i'''}$, in which $v^{1'''}$ is parallel to the particle velocity, by means of Euler angles (φ, θ, ψ). We first rotate v^1 by φ around the axis v^3. Then we rotate $v^{3'}$ by $\theta = 90°$ around $v^{1'}$ to bring $v^{3'}$ into the plane formed by v^1, v^2. Finally we rotate $v^{1''}$ by ψ around $v^{3''}$ to achieve overlapping between $v^{1'''}$ and the velocity $\boldsymbol{v}$. The dependence of φ and ψ on the velocity components v^i are given in (35).

velocity to an arbitrary direction [17]. Let $R(\varphi, \theta, \psi)$ be the transformation which rotates from a general direction to the direction parallel to the velocity which we have denoted v^1; the inverse transformation is R^{-1}. The Euler angles (φ, θ, ψ) are given by

$$\sin\varphi = \frac{v^2}{[(v^1)^2+(v^2)^2]^{1/2}} \qquad \cos\varphi = \frac{v^1}{[(v^1)^2+(v^2)^2]^{1/2}} \tag{35a}$$

$$\theta = 90° \tag{35b}$$

$$\sin\psi = \frac{v^3}{[(v^1)^2+(v^2)^2+(v^3)^2]^{1/2}} \qquad \cos\psi = \frac{[(v^1)^2+(v^2)^2]^{1/2}}{[(v^1)^2+(v^2)^2+(v^3)^2]^{1/2}}. \tag{35c}$$

The general expressions for the transport coefficients are

$$Q^{\nu\mu} = g^{\nu\mu} = R^{-1}(\hat{Q}^{\nu\mu})R = (Q_{\parallel} - Q_{\perp})v^{\nu}v^{\mu}/v^2 + Q_{\perp}\delta^{\nu\mu} \tag{36}$$

$$Q_{\nu\mu} = g_{\nu\mu} = R^{-1}(\hat{Q}_{\nu\mu})R = \frac{(Q_{\parallel}^{-1} - Q_{\perp}^{-1})v^{\nu}v^{\mu}}{v^2} + Q_{\perp}^{-1}\delta_{\nu\mu} \tag{37}$$

and

$$K^{\nu} = R^{-1} \qquad \eta(v)\delta^{1\nu} = \eta(v)v^{\nu}/v \tag{38}$$

where $\hat{Q}_{..}$, $\hat{Q}^{..}$ denote the diffusion tensor in the diagonal system. In view of the high symmetry, the calculation of the connections leads to only three distinct Christoffel symbols in the diagonal system,

$$\Gamma^1_{11} = -\tfrac{1}{2}(\ln Q_{\parallel})' \tag{39a}$$

$$\Gamma^1_{22} = \Gamma^1_{33} = -\tfrac{1}{2}Q_{\parallel}(Q_{\perp}^{-1})' + Q_{\parallel}(Q_{\parallel}^{-1} - Q_{\perp}^{-1})/v \tag{39b}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^1_{13} = \Gamma^1_{31} = -\tfrac{1}{2}(\ln Q_{\perp})' \tag{39c}$$

where the prime denotes a derivative with respect to v. From these relations we are able to calculate the scalar curvature R, the covariant divergence $\tilde{h}^{\nu}_{;\nu}$ and the other functions appearing in the short-time propagator.

Since the short-time propagator is much simplified when expressed in normal coordinates at the postpoint $\boldsymbol{v}_{\mathrm{p}} = \boldsymbol{v}$, we shall also describe briefly the details of this transformation. The prepoint $\boldsymbol{v}_0$ is now referred to as normal coordinate $v_0^{\nu} \to {}^*v^{\nu}$. The general formula for transforming to normal coordinates which are flat at the postpoint is given by†

$$v_{\mathrm{p}}^{\nu} - v_0^{\nu} = v_{\mathrm{p}}^{\nu} - {}^*v_0^{\nu} + \tfrac{1}{2}\Gamma^{\nu}_{\mu\kappa}(\boldsymbol{v}_{\mathrm{p}})({}^*v_0^{\mu} - v_{\mathrm{p}}^{\mu})({}^*v_0^{\kappa} - v_{\mathrm{p}}^{\kappa}). \tag{40}$$

If we choose cartesian coordinates in which v^1 is directed parallel to the postpoint velocity $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{p}}$, we get the following form of (4) for the case of a locally isotropic background:

$$\begin{aligned} v_{\mathrm{p}}^1 - v_0^1 &= v_{\mathrm{p}}^1 - {}^*v^1 + \tfrac{1}{2}\Gamma^1_{11}({}^*v^1 - v_{\mathrm{p}}^1)^2 + \tfrac{1}{2}\Gamma^1_{22}({}^*v^2 - v_{\mathrm{p}}^2)^2 + \tfrac{1}{2}\Gamma^1_{33}({}^*v^3 - v_{\mathrm{p}}^3)^2 \\ &\quad + ({}^*v^1 - v_{\mathrm{p}}^1)[\Gamma^1_{12}({}^*v^2 - v_{\mathrm{p}}^2) + \Gamma^1_{13}({}^*v^3 - v_{\mathrm{p}}^3)] \\ v_0^2 &= {}^*v^2 \\ v_0^3 &= {}^*v^3. \end{aligned} \tag{41}$$

† See, for example [16], pp 19, 20.

The derivatives of the old coordinates $\boldsymbol{v}_0$ relative to the new ones $^*\boldsymbol{v}$ are given by

$$\begin{aligned}
&\partial v_0^1/\partial^* v^1 = 1 - \Gamma^1_{11}{}^*\eta^1 - \Gamma^1_{21}({}^*\eta^2 + {}^*\eta^3) = a \\
&\partial v_0^1/\partial^* v^2 = -\Gamma^1_{22}{}^*\eta^2 - \Gamma^1_{21}{}^*\eta^1 = b \\
&\partial v_0^1/\partial^* v^3 = -\Gamma^1_{13}{}^*\eta^3 - \Gamma^1_{33}{}^*\eta^1 = c \\
&\partial v_0^2/\partial^* v^i = \delta^2_i \qquad \partial v_0^3/\partial^* v^i = \delta^3_i
\end{aligned} \tag{42}$$

where $^*\boldsymbol{\eta} = \boldsymbol{v}_p - {}^*\boldsymbol{v}$.

According to (42), the differentials in the two coordinate systems $\boldsymbol{v}_0, {}^*\boldsymbol{v}$ relate to one another according to the following linear relations:

$$\begin{bmatrix} \delta v_0^1 \\ \delta v_0^2 \\ \delta v_0^3 \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta^* v^1 \\ \delta^* v^2 \\ \delta^* v^3 \end{bmatrix} \qquad \begin{bmatrix} \delta^* v^1 \\ \delta^* v^2 \\ \delta^* v^3 \end{bmatrix} = \begin{bmatrix} a^{-1} & -b/a & -c/a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta v_0^1 \\ \delta v_0^2 \\ \delta v_0^3 \end{bmatrix}. \tag{43}$$

At the postpoint $\boldsymbol{v}_p$, the differential $^*\boldsymbol{\eta}$ vanishes, so that at that point $\partial v_0^\nu/\partial^* v^\mu = \partial^* v^\nu/\partial v_0^\mu = \delta^\nu_\mu$. We thus conclude that we can substitute in the expression for the propagator in normal coordinates, equation (26), the values of the various original functions calculated at the point $\boldsymbol{v}_p$.

We also need the Jacobian of this transformation for finding the new distribution function $^*f(\boldsymbol{r}, {}^*\boldsymbol{v})$ at the prepoint. From (43) we obtain

$$J = |\partial \boldsymbol{v}_0/\partial^* \boldsymbol{v}| = \partial v_0^1/\partial^* v^1 = 1 + \Gamma^1_{11}{}^*\eta^1 + \Gamma^1_{12}{}^*\eta^2 + \Gamma^1_{33}{}^*\eta^3. \tag{44}$$

Using the conservation of the number of particles in a given infinitesimal volume element $f(\boldsymbol{v}_0)\,\mathrm{d}\boldsymbol{v}_0 = {}^*f({}^*\boldsymbol{v})\,\mathrm{d}^*\boldsymbol{v} = f({}^*\boldsymbol{v})J({}^*\boldsymbol{v})\,\mathrm{d}^*\boldsymbol{v}$, we find the new distribution function $^*f({}^*\boldsymbol{v})$ to be

$$^*f({}^*\boldsymbol{v}) = f({}^*\boldsymbol{v})J({}^*\boldsymbol{v}). \tag{45}$$

4. The condition for equilibrium using the short-time transition probability

We will now use the results of §§ 2 and 3 to obtain the equilibrium condition under which the equilibrium distribution preserves its functional form under the operation of the short-time transition probability. According to (29), using normal coordinates, the short-time transition probability in this section will have the following form:

$$P(\boldsymbol{r}, \boldsymbol{v}; \boldsymbol{r}_0, {}^*\boldsymbol{v}_0, \tau) = P_v(\boldsymbol{r}_0, \boldsymbol{v}; {}^*\boldsymbol{v}_0, \tau)\delta(\boldsymbol{r} - \boldsymbol{r}_0 - \boldsymbol{v}\tau). \tag{46}$$

The transformation to normal coordinates at the postpoint does not affect the final velocity $\boldsymbol{v} = \boldsymbol{v}_p$, so that the spatial transition probability is not affected.

If the distribution function $^*f(\boldsymbol{r}_0, \boldsymbol{v}_0)$ at the initial time is given, it would take the form $^*f(\boldsymbol{r}, \boldsymbol{v}, \tau)$ at a later short time τ according to the following equation:

$$^*f(\boldsymbol{r}, \boldsymbol{v}, \tau) = \int\!\!\int {}^*P_v(\boldsymbol{r}_0, \boldsymbol{v}; {}^*\boldsymbol{v}_0, \tau)\delta(\boldsymbol{r} - \boldsymbol{r}_0 - \boldsymbol{v}\tau)f(\boldsymbol{r}_0{}^*, \boldsymbol{v}_0)\,\mathrm{d}\boldsymbol{r}_0\,\mathrm{d}^*\boldsymbol{v}_0. \tag{47}$$

We shall first carry out the spatial integration, so that we get

$$^*f(\boldsymbol{r}, \boldsymbol{v}, \tau) = \int {}^*P_v(\boldsymbol{r} - \boldsymbol{v}\tau, \boldsymbol{v}; {}^*\boldsymbol{v}_0, \tau){}^*f(\boldsymbol{r} - \boldsymbol{v}\tau, {}^*\boldsymbol{v}_0)\,\mathrm{d}^*\boldsymbol{v}_0. \tag{48}$$

The explicit form of the velocity subspace short-time propagator takes the form (see (26)-(28))

$$
\begin{aligned}
&{}^*P_v(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v}; {}^*\boldsymbol{v}_0, \tau) \\
&\quad = [1 + {}^*C_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})^*\eta^{\alpha *}\eta^{\beta} + \ldots][(2\pi\tau)^{-3} g(\boldsymbol{r}-\boldsymbol{v}\tau)]^{1/2} \\
&\quad \times \exp[(-\tau^* g_{\nu\mu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})({}^*\eta^{\nu}/\tau - {}^*\tilde{h}^{\nu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})) \\
&\quad \times ({}^*\eta^{\mu}/\tau - \tilde{h}^{\mu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})) - \tfrac{1}{2}\tau(\partial^*\tilde{h}^{\nu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})/\partial v^{*\nu} + \tfrac{1}{6}{}^*R(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v}))].
\end{aligned}
\tag{49}
$$

We shall now change the variable of integration into ${}^*\xi = {}^*\boldsymbol{\eta}/\sqrt{\tau} = (\boldsymbol{v} - {}^*\boldsymbol{v}_0)/\sqrt{\tau}$. We will also expand the exponent in the propagator to order τ and also expand the dependence on the spatial coordinates to the same order. After these operations have been carried out, (48) takes the form

$$
\begin{aligned}
{}^*f(\boldsymbol{r}, \boldsymbol{v}, \tau) = \int & \frac{\exp[-\frac{1}{2}{}^*g_{\nu\mu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})^*\xi^{\nu *}\xi^{\mu}]}{[(2\pi)^{3*}g(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})]^{1/2}} [1 + \tau^* C_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})^*\xi^{\alpha *}\xi^{\beta} + \ldots] \\
& \times [1 + \tau^* C_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})^*\xi^{\alpha *}\xi^{\beta} + \ldots] \\
& \times [1 + \sqrt{\tau}{}^* g_{\nu\mu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})\tilde{h}^{\mu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})^*\xi^{\nu} \\
& - \tfrac{1}{2}\tau(\partial^*\tilde{h}^{\nu}(\boldsymbol{r}-\boldsymbol{v}\tau, \boldsymbol{v})/\partial^* v^{\nu} + \tfrac{1}{6}{}^*R)] \\
& \times [{}^*f(\boldsymbol{r}, {}^*\boldsymbol{v}_0) - \boldsymbol{v}\tau\partial f(\boldsymbol{r}, \boldsymbol{v}_0)/\partial\boldsymbol{r}].
\end{aligned}
\tag{50}
$$

We shall now substitute the equilibrium Maxwell-Boltzmann distribution function in normal coordinates. In cartesian coordinates in velocity space with v^1 taken parallel to the velocity direction, the phase space distribution takes the form

$$
f_{\mathrm{eq}}(\boldsymbol{r}, \boldsymbol{v}) = N \exp[-\tfrac{1}{2}\beta \boldsymbol{v}^2 - \beta\varphi(\boldsymbol{r})]
\tag{51}
$$

where N is a normalisation constant and $\varphi(\boldsymbol{r})$ is the external potential (including any self-consistent potential).

In normal coordinates (cf (45)) the equilibrium phase space distribution function is expressed as follows at the prepoint $(\boldsymbol{r}, {}^*\boldsymbol{v}_0)$:

$$
\begin{aligned}
{}^*f(\boldsymbol{r}, {}^*\boldsymbol{v}_0) = {} & f_{\mathrm{eq}}(\boldsymbol{r}, {}^*\boldsymbol{v}_0) J(\boldsymbol{r}, {}^*\boldsymbol{v}) N \exp(-\beta\varphi(\boldsymbol{r})) \\
& \times \exp\{-\tfrac{1}{2}\beta[v^1 - {}^*\eta^1 - \tfrac{1}{2}\Gamma^1_{11}(\boldsymbol{r}, \boldsymbol{v})({}^*\eta^1)^2 - \tfrac{1}{2}\Gamma^1_{22}(\boldsymbol{r}, \boldsymbol{v})({}^*\eta^2)^2 \\
& - \tfrac{1}{2}\Gamma^1_{33}(\boldsymbol{r}, \boldsymbol{v})({}^*\eta^3)^2 - \Gamma^1_{21}(\boldsymbol{r}, \boldsymbol{v})^*\eta^{1*}\eta^2 - \Gamma^1_{31}(\boldsymbol{r}, \boldsymbol{v})^*\eta^{1*}\eta^3]^2 \\
& - \tfrac{1}{2}\beta(({}^*\eta^2)^2 + ({}^*\eta^3)^2)][1 + \Gamma^1_{11}(\boldsymbol{r}, \boldsymbol{v})^*\eta^1 + \Gamma^1_{21}(\boldsymbol{r}, \boldsymbol{v})^*\eta^2 + \Gamma^1_{31}{}^*\eta^3]
\end{aligned}
\tag{52}
$$

where ${}^*\boldsymbol{\eta} = \boldsymbol{v} - {}^*\boldsymbol{v}_0$. Putting this expression in (50), replacing ${}^*\boldsymbol{\eta}$ by ${}^*\boldsymbol{\xi}$ as a variable, and expanding the remaining spatial dependence around $\boldsymbol{r}$ and then carrying out the integrations, we obtain

$$
{}^*f(\boldsymbol{r}, \boldsymbol{v}, \tau) = N \exp[-\tfrac{1}{2}\beta(v^1)^2 - \beta\varphi(\boldsymbol{r})][1 + \tau F_A + \tau F_B + \mathrm{O}(\tau^2)]
\tag{53}
$$

where

$$
\begin{aligned}
F_A = {} & h^1(\beta v^1 + \Gamma^1_{11}) + \tfrac{1}{2}\beta^2(v^1)^2 g^{11} - \tfrac{1}{2}\beta(g^{11} + g^{22} + g^{33}) \\
& + \beta v^1 g^{11}\Gamma^1_{11} + \tfrac{1}{2}\beta v^1(g^{11}\Gamma^1_{11} + g^{22}\Gamma^1_{22} + g^{33}\Gamma^1_{31}) - \partial^* h^{\nu}/\partial^* v^{\nu} - \tfrac{1}{6}{}^*R
\end{aligned}
\tag{54a}
$$

and

$$
F_B = B^1\beta v^1 + (\partial\varphi/\partial\boldsymbol{r})\beta\boldsymbol{v}.
\tag{54b}
$$

The equilibrium condition is the vanishing of the term linear in τ.

The local force $\boldsymbol{B}(\boldsymbol{r})$ is derived from the potential: $\boldsymbol{B}(\boldsymbol{r}) = -\partial\varphi/\partial\boldsymbol{r}$, from which we see that the term F_B vanishes. The two terms of F_B correspond to the terms $B(\boldsymbol{r})\partial f/\partial\boldsymbol{v} + \boldsymbol{v}\partial f/\partial\boldsymbol{r}$ in the BFPE. We expect equilibrium to be conserved also in the absence of collisions.

It is also considered that the addition of the $\boldsymbol{B}(\boldsymbol{r})$ to the drift and the introduction of $\delta(\boldsymbol{r}-\boldsymbol{r}_0-\boldsymbol{v}\tau)$ into the short-time transition probability lead exactly to these terms in the Boltzmann equation, although the more general propagator in velocity subspace has been introduced. Thus we conclude that extending the propagator from velocity to phase space in a way analogous to Chandrasekhar is found to be justified.

We also require that the term labelled F_A in (54*a*) to vanish as well. The vanishing of this term is peculiar to the covariant formulation of DG. Actually, it is also the covariant condition for retaining equilibrium in the reduced velocity subspace. In the next section, it will be shown that the condition $F_A = 0$ is equivalent to the analogous condition in the framework of the covariant FP equation. It should also be clarified that the condition $F_A = 0$ does not contain a meaningless relation between quantities of different types but instead relates the values of some specific components calculated at the postpoint with a specific frame of reference.

5. The covariant FP equation and the equilibrium condition

Using $Q_{\nu\mu}$ as the metric $g_{\nu\mu}$, Graham also formulated a covariant FP equation [5]. In this section we shall briefly analyse the conditions for equilibrium in the framework of this formalism. We show that the condition that we obtain is equivalent to the one we determined in the previous section. We will also demonstrate the equivalence with the condi on for equilibrium obtained using a non-covariant equation. We then obtain a straightforward generalisation of Chandrasekhar's relation showing a proportionality even in the general case between the diffusion and the covariant drift for the case of equilibrium.

For a given FP equation

$$\frac{\partial f(\boldsymbol{v})}{\partial t} = -\frac{\partial(K^\nu f(\boldsymbol{v}))}{\partial v^\nu} + \frac{1}{2}\frac{\partial^2(Q^{\nu\mu}f(\boldsymbol{v}))}{\delta v^\nu\,\partial v^\mu} \tag{55}$$

one can formulate the covariant FP equation on the basis of a covariant length given by

$$\mathrm{d}s^2 = g_{\nu\mu}\,\mathrm{d}v^\nu\,\mathrm{d}v^\mu. \tag{56}$$

Graham has shown that the diffusion matrix $Q^{\nu\mu}(\boldsymbol{v})$ transforms like a tensor under coordinate transformations with $g_{\nu\mu}$ as the metric, but the drift $K^\nu(\boldsymbol{v})$ does not transform like a vector and must be replaced by a covariant drift h^ν defined by

$$h^\nu = K^\nu - \frac{1}{2\sqrt{g}}\frac{\partial(\sqrt{g}g^{\nu\mu})}{\partial v^\mu}. \tag{57}$$

With Graham's choice of metric, the distribution function and transition probability are scalar densities rather than scalars and hence we can write them in the form

$$f = \sqrt{g}S \tag{58}$$

where S is the corresponding scalar distribution function. With the definition of a covariant volume element:

$$\mathrm{d}\Omega = \sqrt{g}\,\mathrm{d}\boldsymbol{v} \tag{59}$$

we observe that $f\,\mathrm{d}\boldsymbol{v} = S\,\mathrm{d}\Omega$, which transforms like a scalar under coordinate transformations to conserve the number of particles in a certain infinitesimal volume element.

If in (55) K^ν is expressed in terms of h^ν and the derivatives with respect to v are transformed into covariant ones, we obtain the covariant FP equation:

$$\dot{S} = -[h^\nu S - \tfrac{1}{2} g^{\nu\mu} S_{;\mu}]_{;\nu} \equiv -F^\nu{}_{;\nu} \tag{60}$$

where F^ν is the covariant probability current.

Let us now proceed to determine the conditions for equilibrium. The covariant distribution function is conserved when the divergence of the probability current vanishes:

$$\dot{S} \equiv -F^\nu{}_{;\nu} \equiv 0. \tag{61}$$

For stationary solutions, non-zero probability current is possible although the covariant divergence must vanish; however, for the static solutions corresponding to thermodynamic equilibrium, we expect the probability current F^ν to vanish everywhere. Since S is a scalar, $S_{;\mu} = S_{,\mu}$ and the condition for the vanishing of F^ν can be written in the form

$$F^\nu \equiv h^\nu S - \tfrac{1}{2} g^{\nu\mu} (S)_{,\mu} \equiv 0. \tag{62}$$

After dividing by S†, we obtain the equilibrium condition:

$$h^\nu - \tfrac{1}{2} g^{\nu\mu} (\ln S)_{,\mu} \equiv 0. \tag{63}$$

Substituting in (60) the Maxwell distribution function $f(\boldsymbol{v}) = N \exp(-\beta \boldsymbol{v}^2)$, after some lengthy but straightforward algebra, using the specific properties given § 3, we find identical conditions to those we obtained in the last section (the vanishing of F_A defined in (54a)):

$$\begin{aligned} &h^1(\beta v^1 + \Gamma^1_{11}) + \tfrac{1}{2}\beta^2 (v^1)^2 g^{11} - \tfrac{1}{2}\beta(g^{11} + g^{22} + g^{33}) \\ &\qquad + \beta v^1 g^{11} \Gamma^1_{11} + \beta v^1 (g^{11}\Gamma^1_{11} + g^{22}\Gamma^1_{22} + g^{33}\Gamma^1_{33}) - h^\nu{}_{;\nu} - \tfrac{1}{6} R = 0. \end{aligned} \tag{64}$$

We thus find identical conditions for equilibrium both by the use of the short-time propagator in normal coordinates and by the covariant FP equation. It is also true that the covariant condition (63) is consistent with the non-covariant condition (cf the appendix)

$$\tilde{F}^\nu \equiv K^\nu f - \frac{1}{2} \frac{\partial (Q^{\nu\mu} f)}{\partial v^\nu} \equiv 0 \tag{65}$$

where $\tilde{F}^\nu$ is the non-covariant probability current. After dividing by f we get the relation

$$K^\nu - \frac{1}{2} \frac{\partial Q^{\nu\mu}}{\partial v^\nu} - \tfrac{1}{2} Q^{\nu\mu} \frac{\partial \ln f}{\partial v^\nu} = 0. \tag{65a}$$

We shall now evaluate (63) for the case of a locally isotropic background in the locally cartesian coordinate system. Using the various relations given in § 3, we obtain the following specific condition for equilibrium:

$$F^\nu \equiv [h(v) - \tfrac{1}{2}(\partial \ln S / \partial v) Q_\parallel(v)](v^\nu / v) S = 0 \tag{66}$$

where

$$h^\nu(\boldsymbol{v}) = h(v) v^\nu / v \tag{67a}$$

† For the case of a Maxwellian distribution function, it happens that division by S is legitimate, since it is never zero except for infinite velocity.

and, working out the details of (18),

$$h(v)=[\eta(v)-\tfrac{1}{2}[2(Q_{\parallel}-Q_{\perp})/v+Q'_{\parallel}]+\tfrac{1}{2}(\ln \tilde{Q})'Q_{\parallel} \tag{67b}$$

then

$$h(v)-\tfrac{1}{2}(\partial(\ln S)/\partial v)Q_{\parallel}=0. \tag{67c}$$

We get the result that, for the most general FP equation, equilibrium in locally isotropic media requires that a direct proportionality relation exists between the scalar drift $h(v)$ and the diffusion parallel to the particle motion. The factor of proportionality is $\frac{1}{2}\partial \ln(S)/\partial v$. The relation in (67) can be considered as a generalisation of Chandrasekhar's condition for equilibrium†. Chandrasekhar's relation was limited to the case where $Q^{\nu\mu}(v)=Q(v)\delta^{\mu\nu}$, and the relation which he obtained was as follows:

$$\eta(v)+\beta v Q(v)=0 \qquad \eta(v)-(\partial(\ln f)/\partial v)Q_{\parallel}=0 \tag{68}$$

(where $\eta(v)$ is negative) while in the non-covariant formulation of the FP equation the condition for the vanishing of the regular drift $\tilde{F}^{\nu}$ (65a) takes the form:

$$\eta(v)-\frac{1}{2}\left(\frac{2(Q_{\parallel}-Q_{\perp})}{v}+\frac{\partial Q_{\parallel}}{\partial v}\right)-\frac{1}{2}\frac{\partial(\ln f)}{\partial v}Q_{\parallel}=0. \tag{69}$$

Due to the presence of the middle term, a simple balance between the drift and the diffusion parallel to the particle velocity $Q_{\parallel}$ does not exist. It is to be emphasised that it is only through the introduction of the covariant drift h^{ν}, which combines the standard drift K^{ν} together with the first derivatives of $Q^{\nu\mu}$, that one is able to obtain such a generalisation of Chandrasekhar's relation‡.

6. Summary and discussion

In this paper we have extended Chandrasekhar's work by introducing a short-time transition probability in phase space consistent with the most general BFPE in cartesian coordinates. We have also studied in detail the case of a locally isotropic background and formulated conditions for equilibrium in a form which extends Chandrasekhar's equilibrium relation which was obtained for more restrictive conditions.

In the expression for the short-time transition probability, we have introduced the short-time propagator of DG in the velocity subspace. However, it must be emphasised that the path integral solution together with its discretisation for short times is not uniquely defined in the literature [10–12]. Proceeding as in § 4, it is easily shown that all of the suggested propagators reproduce the FP equation up to order τ.

The question of uniqueness is beyond the scope of this paper. However, we feel that a few comments are needed for completeness. One aspect relating to the discretisation procedure involves the choice of an arbitrary point between the prepoint $\boldsymbol{v}_0$ and the postpoint $\boldsymbol{v}$ at which the various functions are evaluated [8]. Even when the same selection is made, there remain distinct choices for the short-time propagator. Let us

† See [1], equation (12), p 253.

‡ Chandrasekhar's relation also holds exactly for the case of small-angle two-body scattering due to the inverse square law. It is satisfied there due to the special relation between the drift and the first derivatives of the diffusion tensor unique to that case (see [13]).

consider Wissel's discretisation [10], when evaluated at the postpoint for the short-time propagator:

$$P(\boldsymbol{v};\boldsymbol{v}_0,\tau)=[(2\pi)^3\tilde{Q}(\boldsymbol{v})]^{-1/2}\exp\{-\tfrac{1}{2}\tau Q_{\nu\mu}(\boldsymbol{v})[-Q^{\nu\lambda}{}_{,\lambda}(\boldsymbol{v})+K^{\nu}(\boldsymbol{v})-(v^{\nu}-v_0^{\nu})/\tau]$$
$$\times[-Q^{\mu\rho}{}_{,\rho}(\boldsymbol{v})+K^{\mu}(\boldsymbol{v})-(v^{\mu}-v_0^{\mu})/\tau]-\tfrac{1}{2}Q^{\nu\mu}{}_{,\nu\mu}(\boldsymbol{v})+K^{\nu}{}_{,\nu}(\boldsymbol{v})\} \tag{70}$$

where the commas denote ordinary derivatives. Comparing this formula with the DG short-time propagator in (24), we can see the differences. Wissel's form is not a manifestly covariant formula. More important is the functional form by which the two propagators differ; while Wissel's formula is a quadratic form in $\boldsymbol{\eta}=\boldsymbol{v}-\boldsymbol{v}_0$ in the exponent, that of DG is multiplied by a polynomial of degree six. That means that these propagators assign different transition probabilities in velocity space while still reproducing the same first two averaged moments of the FP equation.

The results obtained in this paper were expressed for a specific choice of coordinates. This limitation is removed in another paper [14]; in that paper we introduce a covariant framework for transforming the BFPE, presenting its short-time transition probability in the new set of coordinates. There we also compare Graham's covariant formulation with that of Rosenbluth *et al* [13], based on the use of the ordinary metric in velocity subspace. Both formulations are shown to yield the same BFPE equation in the new coordinate frame, but Graham's approach is more appropriate for presenting the short-time transition probability for the BFPE.

Our extension of the short-time transition probability, as described in the present work, can also be used as a basis for numerical calculation instead of the approaches based on the formulation of Rosenbluth *et al* which have been extensively exploited†. Building a suitable grid and calculating the transition probabilities between its elements serves to follow the evolution of a test particle distribution function by iterative multiplication of short-time transition probabilities‡. That procedure can replace the finite difference methods of solving the partial differential BFPE. Alternatively one could define a finite interval transition probability and solve the evolution in that form.

Acknowledgments

We would like to thank J R Ipser, J Katz and M F Wehner for useful discussions on various aspects of our work.

Appendix. Equivalence of the covariant and non-covariant equilibrium conditions

Using (18) we can write for h^{ν}:

$$h^{\nu}=K^{\nu}-\frac{1}{2}\sqrt{\tilde{Q}}\,\frac{\partial(\tilde{Q}^{-1/2}Q^{\nu\mu})}{\partial v^{\mu}}$$
$$=K^{\nu}-\frac{1}{2}\frac{\partial Q^{\nu\mu}}{\partial v^{\mu}}+\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\ln Q^{1/2}}{\partial v^{\mu}}. \tag{A1}$$

† Ipser has introduced such a calculation in [16].

‡ Explicit numerical calculations based on the one-dimensional short-time propagator have been carried out by Wehner and Wolfer [18].

From (58) we get for $(\ln S)_{,\mu}$:

$$(\ln S)_{,\mu} = \ln(f\tilde{Q}^{1/2})_{,\mu} = (\ln f)_{,\mu} + (\ln \tilde{Q}^{1/2})_{,\mu}. \tag{A2}$$

Substituting (A1) and (A2) into (63) leads to (65*a*) which proves the equivalence of the two equilibrium conditions:

$$h^{\nu} - \tfrac{1}{2}Q^{\nu\mu}(\ln S)_{,\mu} = K^{\nu} - \frac{1}{2}\frac{\partial Q^{\nu\mu}}{\partial v^{\mu}} - \tfrac{1}{2}Q^{\nu\mu}\frac{\partial(\ln f)}{\partial v^{\mu}} = 0. \tag{A3}$$

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